NORTH-HOLLAND

A Theory of Pseudoskeleton Approximations

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ABSTRACT

Let an $m \times n$ matrix A be approximated by a rank-r matrix with an accuracy ε. We prove that it is possible to choose r columns and r rows of A forming a so-called pseudoskeleton component which approximates A with $\mathscr{O}(\varepsilon\sqrt{r}(\sqrt{m} + \sqrt{n}))$ accuracy in the sense of the 2-norm. On the way to this estimate we study the interconnection between the volume (i.e., the determinant in the absolute value) and the minimal singular value σ_r of $r \times r$ submatrices of an $n \times r$ matrix with orthogonal columns. We propose a lower bound (better than one given by Chandrasekaran and Ipsen and by Hong and Pan) for the maximum of σ_r over all these submatrices and formulate a hypothesis on a tighter bound. © Elsevier Science Inc., 1997

1. INTRODUCTION

Many applications involve coefficient matrices whose blocks can be easily approximated by low-rank matrices. Such block low-rank matrix approximations can be computed, for example, by partial SVD algorithms and may underlie construction of many efficient numerical methods. Partial SVD algorithms typically require all entries of a block to construct a block low-rank approximation. However, we may anticipate dramatical savings in memory

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0024-3795/97/$17.00
PII S0024-3795(96)00301-1

and arithmetic if it were possible to approximate a block using only a small part of its entries. In this paper we present a theory which lays the ground for construction of such algorithms.

Let $A \in \mathbb{R}^{m \times n}$, and assume that

$$\operatorname{rank} A = r. \tag{1.1}$$

Then there exists a nonsingular $r \times r$ submatrix $\hat{A}$ in A. If $\hat{A}$ lies in rows $i \in \hat{I} \equiv \{i_1, \dots, i_r\}$ and in columns $j \in \hat{J} \equiv \{j_1, \dots, j_r\}$, we will write

$$\hat{A} = A(\hat{I}, \hat{J}). \tag{1.2}$$

It is easy to verify that

$$A = C\hat{A}^{-1}R, \tag{1.3}$$

where

$$C = A(I, \hat{J}), \qquad R = A(\hat{I}, J),$$
$$I \equiv \{1, \dots, m\}, \qquad J \equiv \{1, \dots, n\}. \tag{1.4}$$

The decomposition (1.3) is known as a skeleton decomposition of A.

Now let us suppose that rank $A \approx r$ means that rank$(A + E) = r$, where $E \approx 0$ in the sense of a prescribed matrix norm. The exact equality rank $A = r$ implies the exact equality $A = C\hat{A}^{-1}R$, and we wonder if the approximate equality rank $A \approx r$ may imply the approximate equality $A \approx C\hat{A}^{-1}R$.

Using a rather standard technique, we can prove that for ε sufficiently small

$$\|A - C\hat{A}^{-1}R\|_2 = \mathcal{O}\left(\|A\|_2^2 \|\hat{A}^{-1}\|_2^2 \varepsilon\right), \tag{1.5}$$

provided that $\hat{A}$ is nonsingular (see [4]). Consequently, we cannot state in general that $A \approx C\hat{A}^{-1}R$, because $\hat{A}$ may be ill conditioned or even singular.

We can be better off replacing $\hat{A}^{-1}$ with a more suitable square matrix G, thus trying to approximate A by a matrix $B = CGR$ of rank r or less. Any matrix of the form $B = CGR$ will be called a pseudoskeleton component of A.

The main purpose of this paper is to prove that whenever A is approximated by a rank-r matrix with accuracy ε, it can be as well approximated by its pseudoskeleton component with accuracy $\mathcal{O}(\varepsilon\sqrt{r}(\sqrt{m} + \sqrt{n}))$.

The choice of the proper pseudoskeleton component (in other words, the choice of C, R, and G) has much to do with the problem of seeking a submatrix with the best-bounded inverse. This problem is considered in Section 2. Therein we improve the recently proposed nontrivial estimate for the 2-norm value of the inverse of such a submatrix [1, 3]. Our proof is absolutely trivial (and does not need the CS-decomposition arguments). We propose also a hypothesis on a tighter estimate. We close the section with a discussion of the interconnection between the volume (i.e., the determinant in the absolute value) and the minimal singular value σ_r of $r \times r$ submatrices of an $n \times r$ matrix with orthogonal columns.

In Section 3 we derive approximation estimates for the pseudoskeleton decomposition which do not involve the norms of A and of $\hat{A}^{-1}$. Also, these estimates do not use the assumption that ε is sufficiently small. Therefore, these estimates differ from the ones typical of small-perturbations theory. We consider also the case when G may be constructed only from the entries of C and R, specifically from the entries of $\hat{A}$.

In Section 4 we present a numerical example which is chosen to illustrate the approximation capabilities of pseudoskeleton components for a nontrivial application. We consider here a matrix arising from the 3D scattering problem for a perfectly conducting sphere. We note that the pseudoskeleton approximation may require far less memory and arithmetic than the partial SVD techniques. It allows one not to compute all the matrix entries, working instead only with small part of them.

In Section 5, a brief discussion of the results is given.

2. SUBMATRICES WITH THE BEST-BOUNDED INVERSES

From (1.5) we see that a pseudoskeleton component provides an accurate enough approximation if the value of $\|\hat{A}^{-1}\|$ is not very large. An optimization of this norm value by making a proper choice of the columns and the rows can improve the quality of the approximation. However, in order to understand whether such an optimization is feasible, we need *a priori* estimates of the norm value of the submatrix with the best-bounded inverse.

Consider first a model (and apparently the simplest) case

$$A = \begin{bmatrix} U^T \\ 0 \end{bmatrix}, \tag{2.1}$$

where

$$U^T U = I, \qquad U \in \mathbb{R}^{n \times r}, \quad r \leqslant n. \tag{2.2}$$

Denote by $\mathcal{M}(U)$ the set of all $r \times r$ submatrices in U, and set

$$t(r, n) = \frac{1}{\min\limits_{U} \max\limits_{P \in \mathcal{M}(U)} \sigma_{\min}(P)}, \tag{2.3}$$

where $\sigma_{\min}(P)$ denotes the minimal singular value of P.

In order to derive an upper estimate of $t(r, n)$, consider such a submatrix $\hat{P} \in \mathcal{M}(U)$ that has the maximal volume (i.e., the maximal absolute value of the determinant). Due to the Binet-Cauchy formula we have

$$1 = \det(U^T U) = \sum_{P \in \mathcal{M}(U)} (\det P)^2.$$

Since the number of summands equals $C_n^r = n!/[r!(n-r)!]$, we conclude that

$$(\det \hat{P})^2 \geqslant 1/C_n^r.$$

Let $\sigma_1(\hat{P}) \geqslant \cdots \geqslant \sigma_r(\hat{P})$ be the singular values of $\hat{P}$. Then

$$(\det \hat{P})^2 = \sigma_1^2(\hat{P}) \ldots \sigma_r^2(\hat{P}) \leqslant \sigma_r^2(\hat{P}) \left[\sigma_1^2(\hat{P})\right]^{r-1} \leqslant \sigma_r^2(\hat{P})$$

for $\sigma_1(\hat{P}) \leqslant 1$. Hence,

$$t(r, n) \leqslant \sigma_r^{-1}(\hat{P}) \leqslant \sqrt{C_n^r}.$$

However (thanks to the results from [1, 3]), a much finer estimate for $\sigma_r(\hat{P})$ can be derived. We present below an even better estimate (with an absolutely trivial proof).

LEMMA 2.1. *$t(r, n)$ defined by (2.3) satisfies*

$$t(r, n) \leqslant \sqrt{r(n-r)+1}. \tag{2.4}$$

Proof. Without loss of generality we may assume that the submatrix $\hat{P}$ of the greatest possible volume resides in the first r rows. It is then evident that the submatrix of maximal volume in the matrix

$$\tilde{U} = U\hat{P}^{-1} = \begin{bmatrix} I \\ V \end{bmatrix}$$

is located in the same r rows. If $|\tilde{u}_{ij}| > 1$, then by swapping the ith and the jth rows we could obtain on the first r rows a submatrix whose volume was greater than 1. This contradicts the choice of $\hat{P}$. We thus have

$$\sigma_{\min}^{-1}(\hat{P}) = \sigma_{\max}(\tilde{U}) \leqslant \sqrt{\|I\|_2^2 + \|V\|_2^2} \leqslant \sqrt{r(n-r)+1},$$

and hence

$$t(r,n) \leqslant \sqrt{r(n-r)+1}. \qquad \blacksquare \tag{2.5}$$

The equality in the estimate of Lemma 2.1 is attained in the two extreme cases: $r = 1$ and $r = n - 1$. We believe that for $1 < r < n - 1$ this estimate is not sharp. We do not know any matrix for which the inequality

$$t(r,n) \leqslant \sqrt{n} \tag{2.6}$$

is violated. However, so far (2.6) is still a hypothesis.

The orthogonal matrix

$$U = \begin{bmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & & & 1/\sqrt{n-r+1} \\ & & & \vdots \\ & 0 & & 1/\sqrt{n-r+1} \end{bmatrix}$$

illustrates that the estimate (2.6) indeed is almost sharp. Moreover, there are infinitely many values of r and n, $1 < r < n - 1$, for which (2.6) cannot be improved.

LEMMA 2.2. *If r is a positive integer and n is divisible by $r + 1$, then*

$$t(r,n) \geqslant \sqrt{n}.$$

Proof. Let $Q \equiv [A \ b] \in \mathbb{R}^{(r+1)\times(r+1)}$ be any orthogonal matrix such that

$$b = \left[\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right]^T.$$

We construct the CS decomposition of Q which corresponds to the diagonal blocks of orders r and 1, and we can easily see that if $P \in \mathbb{R}^{r\times r}$ resides in the first r rows of A then

$$\sigma_{\min}(P) = \frac{1}{\sqrt{n}}.$$

Repeating the same arguments with different permutations of rows of Q, we conclude that the above equality holds true for any $P \in \mathscr{M}(A)$. Now, let

$$U = \frac{1}{\sqrt{k}}\begin{bmatrix} A \\ A \\ \vdots \\ A \end{bmatrix} \in \mathbb{R}^{n\times r}, \qquad n = k(r+1).$$

The required inequality follows from the observation that

$$\max_{P\in\mathscr{M}(U)} \sigma_{\min}(P) = \frac{1}{\sqrt{n}}. \qquad \blacksquare$$

We close the section by noting that the choice of the maximal-volume submatrix ensures only that its minimal singular value is sufficiently large. This minimal singular value is not guaranteed to be the greatest possible one among all submatrices.

For example, consider an $n \times 2$ matrix U of the form

$$U = \begin{bmatrix} 0 & 1 & 1 & \varepsilon^2 & \varepsilon^2 & \cdots & \varepsilon^2 \\ \varepsilon & \varepsilon & 2\varepsilon & -\varepsilon^2 & -\varepsilon^2 & \cdots & -\varepsilon^2 \end{bmatrix}^T.$$

Assume that

$$n > 3,$$

and set

$$\varepsilon = \left(\frac{3}{n-3} \right)^{1/3}.$$

Then the two columns of U are orthogonal.

For large n the greatest possible volume of all 2×2 submatrices of U is equal to ε and there are exactly three such submatrices. These submatrices are located in the first three rows.

After normalizing the columns of U by the scalars d_1 and d_2, we get the orthogonal matrix $\tilde{U}$ in which the same three submatrices still have the maximal volume. It can be easily seen that their minimal singular values differ in the general case.

For instance, consider the 2×2 submatrices

$$A_1 = \begin{bmatrix} p & \varepsilon d_2 \\ d_1 & \varepsilon d_2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & \varepsilon d_2 \\ d_1 & 2\varepsilon d_2 \end{bmatrix}.$$

For any $d_2 > 0$ the minimal singular value of A_1 is greater than that of A_2.

Obviously, we can construct small perturbations of a special form which preserve the norms and the orthogonality of the two columns of $\tilde{U}$, do not change the entries of A_2, and decrease the volume of A_1. Sufficiently small perturbations will preserve as well the inequality between minimal singular values of submatrices A_1 and A_2 while the three submatrices considered above still contain the submatrix of maximal volume.

3. MAIN RESULTS

We now realize that a matrix A of the form

$$A = \begin{bmatrix} U^T \\ 0 \end{bmatrix}, \qquad U^T U = I, \quad U \in \mathbb{R}^{n \times (r-1)},$$

has an $(r-1) \times (r-1)$ submatrix with a sufficiently well-bounded inverse. In the general case the situation may get much worse; for example, consider the matrix

$$A = \begin{bmatrix} & & U^T & \\ \varepsilon & 0 & \cdots & 0 \\ & & \mathbf{0} & \end{bmatrix}, \qquad U^T U = I, \quad U \in \mathbb{R}^{n \times (r-1)}.$$

The minimal singular value of the unique nonsingular submatrix of order r is not greater than ε. We want to estimate the accuracy of pseudoskeleton components with r columns and r rows for an ε-perturbation of A. We obviously come to nothing here when using (1.5). Nevertheless, certain approximation properties of pseudoskeleton components hold true even in this case.

THEOREM 3.1. *Assume that* $A, F \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A - F) \leqslant r$, *and* $\|F\|_2 \leqslant \varepsilon$ *for some* $\varepsilon > 0$. *Then there exist r columns and r rows which determine a pseudoskeleton component CGR such that*

$$\|A - CGR\|_2 \leqslant \varepsilon\left\{1 + \left[\sqrt{t(r,n)} + \sqrt{t(r,m)}\right]^2\right\}. \tag{3.1}$$

Proof. Consider the decomposition

$$A - F = U\Sigma V, \tag{3.2}$$

where $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$, $\sigma_1 \geqslant \cdots \geqslant \sigma_r \geqslant 0$, $U^TU = VV^T = I$, and submatrices $\hat{U}, \hat{V} \in \mathbb{R}^{r\times r}$ of U, V, respectively, such that

$$\|\hat{U}^{-1}\|_2 \leqslant t(r,m), \tag{3.3}$$

$$\|\hat{V}^{-1}\|_2 \leqslant t(r,n). \tag{3.4}$$

We now select the r rows and r columns determined by the choice of $\hat{U}$ and $\hat{V}$, respectively.

Let C and F_C denote $m \times r$ submatrices, and R and F_R denote $r \times n$ submatrices, of A and of F, respectively, which correspond to the selected rows and columns. Let $\hat{A}$ and $\hat{F}$ denote the $r \times r$ submatrices which occupy the intersections of these rows and columns in A and F. Then any pseudoskeleton component CGR, $G \in \mathbb{R}^{r\times r}$, can be presented in the following form:

$$\begin{aligned} CGR &= \left(U\Sigma\hat{V} + F_C\right)G\left(\hat{U}\Sigma V + F_R\right) \\ &= U\Sigma\hat{V}G\hat{U}\Sigma V + E, \end{aligned} \tag{3.5}$$

$$\begin{aligned} E &= \left(U\Sigma\hat{V} + F_C\right)GF_R + F_CG\left(\hat{U}\Sigma V + F_R\right) - F_CGF_R \\ &= U\hat{U}^{-1}(\hat{U}\Sigma\hat{V}G)F_R + F_C(G\hat{U}\Sigma\hat{V})\hat{V}^{-1}V + F_CGF_R \\ &= U\hat{U}^{-1}(\Phi G)F_R + F_C(G\Phi)\hat{V}^{-1}V + F_CGF_R, \end{aligned} \tag{3.6}$$

where

$$\Phi = \hat{U}\Sigma\hat{V} = \hat{A} - \hat{F}. \tag{3.7}$$

We also rewrite (3.2) and (3.5) in terms of Φ:

$$A - F = U\hat{U}^{-1}\Phi\hat{V}^{-1}V, \tag{3.8}$$

$$CGR = U\hat{U}^{-1}(\Phi G\Phi)\hat{V}^{-1}V + E. \tag{3.9}$$

Now consider the singular-value decomposition of Φ:

$$\Phi = \tilde{U}\tilde{\Sigma}\tilde{V}, \qquad \tilde{\Sigma} = \operatorname{diag}(\tilde{\sigma}_1, \dots, \tilde{\sigma}_r), \tilde{U}^T\tilde{U} = \hat{V}^T\hat{V} = I. \tag{3.10}$$

Let $\tau > 0$ be a threshold value which will be specified later. Introducing the notation

$$\tilde{\Sigma}_\tau \equiv \operatorname{diag}(\tilde{\sigma}_{\tau i}), \qquad \tilde{\sigma}_{\tau i} = \begin{cases} \sigma_i & \text{if} \quad \sigma_i \geqslant \tau, \\ 0 & \text{otherwise}, \end{cases} \tag{3.11}$$

$$\tilde{\Sigma}_\tau^+ \equiv \operatorname{diag}(\tilde{\sigma}_{\tau i}^+), \qquad \tilde{\sigma}_{\tau i}^+ = \begin{cases} \sigma_i^{-1} & \text{if} \quad \sigma_i \geqslant \tau, \\ 0 & \text{otherwise}, \end{cases} \tag{3.12}$$

$$\Phi_\tau = \tilde{U}\tilde{\Sigma}_\tau\tilde{V}, \tag{3.13}$$

$$\Phi_\tau^+ = \tilde{V}^T\tilde{\Sigma}_\tau^+\tilde{U}^T, \tag{3.14}$$

we see that

$$\Phi\Phi_\tau^+\Phi = \Phi_\tau, \tag{3.15}$$

and moreover,

$$\|\Phi\Phi_\tau^+\|_2 \leqslant 1, \qquad \|\Phi_\tau^+\Phi\|_2 \leqslant 1. \tag{3.16}$$

It we set

$$G = \Phi_\tau^+, \tag{3.17}$$

then (3.6) and (3.15) imply that

$$\|E\|_2 \leqslant \varepsilon\left(\|\hat{U}^{-1}\|_2 + \|\hat{V}^{-1}\|_2 + \frac{\varepsilon}{\tau}\right). \tag{3.18}$$

Using this inequality in conjunction with (3.8), (3.9), and (3.15), we get the estimate

$$\|A - CGR\|_2 \leqslant \varepsilon + \tau\|\hat{U}^{-1}\|_2\|\hat{V}^{-1}\|_2 + \frac{\varepsilon^2}{\tau} + \varepsilon\|\hat{U}^{-1}\|_2 + \varepsilon\|\hat{V}^{-1}\|_2. \tag{3.19}$$

Now setting $\tau = \varepsilon / \sqrt{\|\hat{U}^{-1}\|_2\|\hat{V}^{-1}\|_2}$, we complete the proof of the theorem. ■

Note that in this proof the singular-value decomposition of Φ is not necessary; we can get the same estimate by applying the "filtering" technique (3.10)–(3.16) directly to the decomposition $\Phi = \hat{U}\Sigma\hat{V}$. However, in that case we must take

$$\tau = \varepsilon\sqrt{\|\hat{U}^{-1}\|_2\|\hat{V}^{-1}\|_2}\,.$$

COROLLARY 3.1. *Under the hypotheses of Theorem* 3.1 *there exists a pseudoskeleton component such that*

$$\|A - CGR\|_2 \leqslant \varepsilon\left(1 + 2\sqrt{rn} + 2\sqrt{rm}\right). \tag{3.20}$$

This bound immediately follows from (3.1) and the above estimates for $t(r, n)$ (see Section 2).

REMARK 3.1. We would like to emphasize that Theorem 3.1 is somewhat different from theorems of small-perturbations theory, where ε must be sufficiently small (for a rather standard result of this type, see [4, Theorem 3.1]). However, in the most interesting and important cases ε may depend on m and n, and usually we are interested in construction of pseudoskeleton approximations for a family of matrices generated by one application.

We believe that the estimate (3.1) is sharp up to a numerical constant. The following example provides the illustration for the case $r = 1$, $n = m$.

EXAMPLE. Let an $n \times n$ matrix A be given by its singular-value decomposition

$$A \equiv U\Sigma V = u_1\sqrt{\varepsilon}v_1 + U_2(\varepsilon I)V_2,$$

where

$$\begin{bmatrix} u_1 & U_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1 \\ V_2 \end{bmatrix}$$

are orthogonal $n \times n$ matrices, $u_1 \in \mathbb{R}^{n\times 1}$, $v_1 \in \mathbb{R}^{1\times n}$, and ε is a positive number.

Let $v_1 = [1/\sqrt{n}, \dots, 1/\sqrt{n}]$. Then for any column c_i of A,

$$(c_i, u_1) = (Ae_i, u_1) = (\Sigma V e_i, e_1) = \sigma_1(e_i, V^T e_1) = \sqrt{\varepsilon}(e_i, v_1^T) = \sqrt{\frac{\varepsilon}{n}},$$

and similarly

$$(c_i, u_k) = \varepsilon(v_k)_i, \qquad 1 < k \leqslant n, \tag{3.21}$$

where u_k and v_k denote columns of U and rows of V, respectively.

By orthogonal invariance of the 2-norm for any pseudoskeleton component $c_i g r_j$ we have the inequality

$$\begin{aligned}\|u_1\sqrt{\varepsilon}v_1 - c_i g r_j\|_2 &= \left\| \begin{bmatrix} \sqrt{\varepsilon} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{\varepsilon}/\sqrt{n} \\ U_2^T c_i \end{bmatrix} g \begin{bmatrix} r_j v_1^T & r_j V_2^T \end{bmatrix} \right\|_2 \\ &\geqslant \left\| \begin{bmatrix} \sqrt{\varepsilon} \\ 0 \end{bmatrix} - g r_j v_1^T \begin{bmatrix} \sqrt{\varepsilon}/\sqrt{n} \\ U_2^T c_i \end{bmatrix} \right\|_2 .\end{aligned} \tag{3.22}$$

The equality (3.21) implies that

$$\|U_2^T c_i\|_2^2 = \varepsilon^2\left(1 - \frac{1}{n}\right),$$

and the right-hand side of the inequality (3.22) can be estimated from below by $\sqrt{f(\hat{g})}$, where

$$f(\hat{g}) = \left(\sqrt{\varepsilon} - \hat{g}\frac{\sqrt{\varepsilon}}{\sqrt{n}}\right)^2 + \hat{g}^2\varepsilon^2\left(1 - \frac{1}{n}\right), \qquad \hat{g} \equiv gr_j v_1^T.$$

Notice that $f(\hat{g})$ is a quadratic in the function $\hat{g}$ and can be readily minimized with the bound

$$\|u_1\sqrt{\varepsilon}\, v_1 - c_i g r_j\|_2 \geqslant \sqrt{\varepsilon - \frac{\varepsilon}{1 + \varepsilon(n-1)}}\,. \tag{3.23}$$

Suppose now that $(n-1)\varepsilon = 1$. Then using (3.23) and the triangle inequality

$$\|u_1\sqrt{\varepsilon}\, v_1 - c_i g r_j\|_2 \leqslant \|u_1\sqrt{\varepsilon}\, v_1 - A\|_2 + \|A - c_i g r_j\|_2,$$

we get the estimate

$$\|A - c_i g r_j\|_2 \geqslant \sqrt{\frac{\varepsilon}{2}} - \varepsilon = \sqrt{\frac{\varepsilon}{2}}\,(1 - \sqrt{2\varepsilon}) \tag{3.24}$$

for any pseudoskeleton component $c_i g r_j$.

Note that this example can be easily extended to the case $r > 1$.

Thanks to Theorem 3.1, we know that accurate enough pseudoskeleton approximations do exist. We also note that our proof is almost constructive and involves two stages:

(1) the choice of appropriate C and R;
(2) the choice of G.

Both stages of the proof make use of the explicit knowledge of F. The first stage consists indeed in looking in $A - F$ for an $r \times r$ submatrix of the maximal volume. The second stage is something like the pseudoinversion procedure applied to $\hat{A} - \hat{F}$ [see (3.7) and the remainder of the proof].

We now want to exploit another choice of G with no explicit information on $\hat{F}$.

THEOREM 3.2. *Assume that C and R are chosen in the same way as in the proof of Theorem 3.1. Then G can be chosen using only $\hat{A}$, and this choice guarantees the accuracy*

$$\|A + CGR\|_2 \leqslant \varepsilon[2 + 2t(r,n) + 2t(r,m) + 5t(r,n)t(r,m)]. \quad (3.25)$$

Proof. Consider the singular-value decomposition

$$\hat{A} = \tilde{U}\tilde{\Sigma}\tilde{V}, \qquad \tilde{\Sigma} = \operatorname{diag}(\tilde{\sigma}_1, \dots, \tilde{\sigma}_r), \quad \tilde{U}^T\tilde{U} = \tilde{V}^T\tilde{V} = I,$$

and define $\tilde{\Sigma}_\tau$, $\tilde{\Sigma}_\tau^+$, $\hat{A}_\tau$, $\hat{A}_\tau^+$ by formulas similar to (3.11)–(3.14). We essentially use the relationships (3.6)–(3.9), which hold true for any G, and consider them in the following form:

$$E = U\hat{U}^{-1}(\hat{A}G)F_R + F_C(G\hat{A})\hat{V}^{-1}V + F_C G F_R$$

$$- U\hat{U}^{-1}\hat{F}GF_R - F_C G\hat{F}\hat{V}^{-1}V,$$

$$A - F = U\hat{U}^{-1}\hat{A}\hat{V}^{-1}V - U\hat{U}^{-1}\hat{F}\hat{V}^{-1}V,$$

$$CGR = U\hat{U}^{-1}(\hat{A}G\hat{A})\hat{V}^{-1}V + E$$

$$- U\hat{U}^{-1}(\hat{F}G\hat{A})\hat{V}^{-1}V - U\hat{U}^{-1}(\hat{A}G\hat{F})\hat{V}^{-1}V$$

$$+ U\hat{U}^{-1}(\hat{F}G\hat{F})\hat{V}^{-1}V.$$

If we set

$$G = \hat{A}_\tau^+,$$

then relations similar to (3.15)–(3.16) are valid:

$$\hat{A}\hat{A}_\tau^+\hat{A} = \hat{A}_\tau,$$

$$\|\hat{A}\hat{A}_\tau^+\|_2 \leqslant 1, \qquad \|\hat{A}_\tau^+\hat{A}\|_2 \leqslant 1.$$

Therefore,

$$\begin{aligned}\|A - CGR\|_2 \leqslant\ & \varepsilon + \tau\|\hat{U}^{-1}\|_2\|\hat{V}^{-1}\|_2 + \varepsilon\|\hat{U}^{-1}\|_2\|\hat{V}^{-1}\|_2 + \varepsilon\|\hat{U}^{-1}\|_2 \\ & + \varepsilon\|\hat{V}^{-1}\|_2 + \frac{\varepsilon^2}{\tau} + \frac{\varepsilon^2}{\tau}\|\hat{U}^{-1}\|_2 + \frac{\varepsilon^2}{\tau}\|\hat{V}^{-1}\|_2 \\ & + 2\varepsilon\|\hat{U}^{-1}\|_2\|\hat{V}^{-1}\|_2 + \frac{\varepsilon^2}{\tau}\|\hat{U}^{-1}\|_2\|\hat{V}^{-1}\|_2.\end{aligned}$$

Taking the threshold value $\tau = \varepsilon$, we derive the estimate (3.25). ■

Note that unlike the proof of Theorem 3.1, the singular-value decomposition of Φ is essential to the statement of Theorem 3.2.

REMARK 3.2. The estimate (3.25) can be improved as follows:

$$\|A - CGR\|_2 \leqslant \varepsilon\sqrt{1 + t^2(r, p)}\left\{1 + \left[\sqrt{t(r, n)} + \sqrt{t(r, m)}\,\right]^2\right\}, \quad (3.26)$$

where $p = \min(m, n)$.

The proof of the inequality (3.26) can be found in [4].

4. THE NUMERICAL EXAMPLE

Here we present a numerical example to illustrate the approximation capabilities of pseudoskeleton components for a nontrivial application. We are making no attempt here to devise really efficient algorithms. The only purpose of the experiments was to demonstrate the existence of an pseudoskeleton component with approximation quality comparable with that of the SVD.

All the constructions were carried out as close to the paths described in the proofs as possible, and all the estimates were obtained in the most conservative way (for example, if we present below an estimate of $\|A\|_2$, it arises from the SVD run).

We have picked a 1048 × 1040 block of a complex matrix of order 20,808 originating from the 3D scattering problem for a perfectly conducting sphere

S. Namely, we solve the so-called electric field integral equation:

$$\left\{ \left(k_0^2 + \operatorname{grad} \operatorname{div} \right) \int_S G(r) \vec{\jmath}(x)\, dx \right\}_\tau = \vec{E}_\tau^0,$$

where

$$G(r) = \frac{e^{ik_0 r}}{r}, \qquad r = |x - y|,$$

τ means the tangential component, $\vec{E}^0$ is proportional to the incident electric field (we used a plane-wave excitation), and $\vec{\jmath}(x)$ is the unknown electric current on the surface S.

The radius of the sphere was 1 m, whereas the wavenumber k_0 of the incident wave was 6.8 m^{-1}.

We have adopted the Galerkin discretization technique in the version proposed by Rao, Wilton, and Glisson [5]. Fragments of the triangular mesh are presented in Figure 1. Each edge matches exactly one unknown in the corresponding linear algebraic system. Shown are the edges corresponding to the chosen matrix block, which had block indices (20, 1) in a 20×20 block partition with blocks of approximately equal sizes. The linear system involved had 20,808 complex unknowns.

The construction of the pseudoskeleton component was based on the SVD of the chosen block $A = U \Sigma V^* = \sum_{i=1}^{n} u_i \sigma_i v_i^*$, computed by the standard LAPACK routine. For a given rank r we defined the matrix F as $F \equiv \sum_{i=r+1}^{n} u_i \sigma_i v_i^*$. To choose r rows and columns which constitute matrices C and R, we applied an algorithm searching for well-conditioned submatrices [1] to the $m \times r$ and $n \times r$ orthonormal matrices $[u_1, \dots, u_r]$ and $[v_1, \dots, v_r]$. The approximation error matrices $A - CGR$ were formed explicitly.

Figures 2 and 3 display the matrix approximation error in the spectral and Frobenius norms for SVD and pseudoskeleton components as a function of their rank r.

The curve labeled "PSA, $G = f(A - F)$" shows the approximation error of the pseudoskeleton component with

$$G = \arg\min_G \| A - CGR \|_F.$$

The Frobenius norm is preferable, since it allows for a complete solution of the minimization problem (see [4]). However, this choice of G is apparently of no practical value, since it uses all elements of the block.

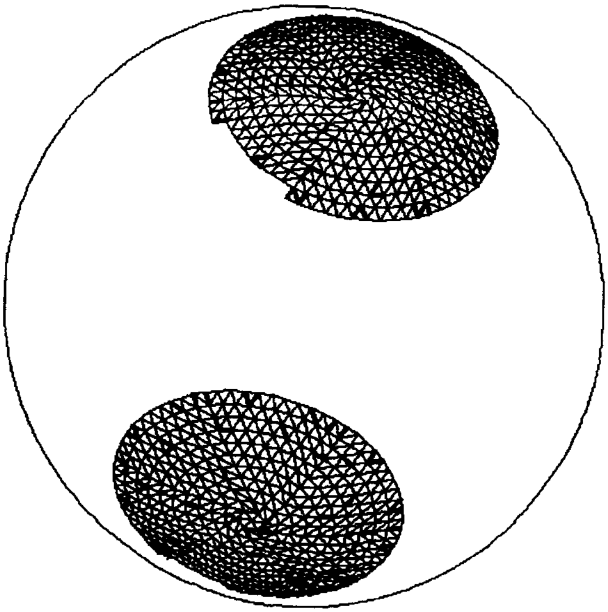

FIG. 1. The mesh fragments corresponding to the chosen block. Highlighted are the edges constituting the well-conditioned 60×60 submatrix found as in the proof of Theorem 3.1.

The curves labeled "PSA, $G = f(\hat{A} - \hat{F})$" and "PSA, $G = f(\hat{A})$" correspond to the choices of G described in the proofs of Theorems 3.1 and 3.2, respectively.

Figures 4 and 5 show the upper bounds of the matrix approximation error provided by Theorems 3.1 and 3.2. The corresponding experimental data are repeated for comparison purposes.

Note that some of the edges depicted in Figure 1 are highlighted. These are edges corresponding to the 60×60 submatrix obtained by a search algorithm of [1] applied to the orthonormal matrices containing 60 senior left and right singular vectors of the chosen block.

In Tables 1 and 2 we present the numerical values of approximation errors displayed in Figures 2–5.

Assume now that we want to approximate the considered matrix block to a 10^{-5} accuracy. From Table 1 we can see the difference between the two following approaches:

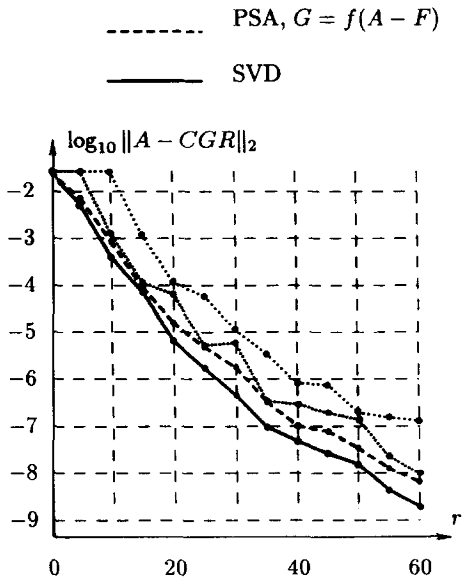

FIG. 2. Approximation error in 2-norm for SVD and pseudoskeleton components.

(1) We compute approximately 10^6 matrix entries and find 20 senior singular triplets of a 1048×1040 matrix.

(2) We compute the entries of 30 rows and columns (which is only 6% of the total number of entries) and find the SVD of 30×30 matrix.

Both approaches provide the same approximation quality, but the second approach may require far less memory and arithmetic. Note that it allows us not to compute all the matrix entries, working instead only with a small part of them.

5. DISCUSSION

The principal conclusion of the above theory can be formulated as follows: the existence of a rank-r approximation generally means that such an approximation can be found only from some r columns and r rows of the matrix to be approximated. If the rank-r approximation has the accuracy ε, then the same (asymptotic) accuracy can be preserved by the pseudoskeleton approximation. Restrictions on the information available for the construction

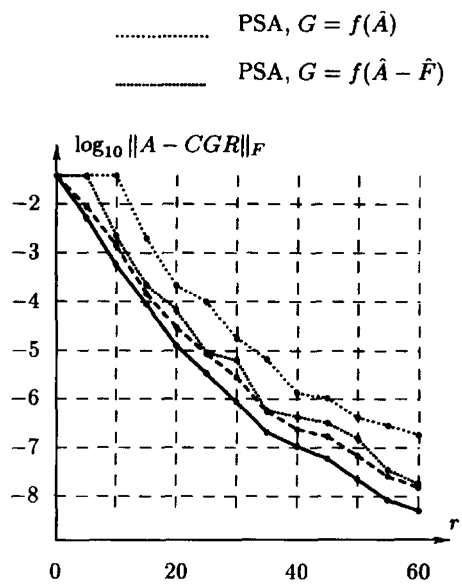

FIG. 3. Approximation error in F-norm for SVD and pseudoskeleton components.

of G degrade the accuracy by a factor of $\mathcal{O}(\sqrt{n})$. Apparently this addresses the worst case. If the original matrix has an $r \times r$ submatrix with sufficiently (for given ε) well-bounded inverse, then it is possible to preserve the same accuracy ε. Moreover, we can expect the same result in the case of an order-r submatrix with well-separated small singular values.

Our concern in this paper has been the estimation of the possible accuracy which can be obtained, in principle, via pseudoskeleton components. Another important matter is how to construct such components in practice, and in particular, how to select appropriate columns and rows. We believe that results of [1] will promote a suitable algorithm.

Some proposals can be derived right away from our estimates. Since the deterioration factor incorporates (Theorem 3.1) the original matrix size, it seems reasonable to partition the matrix into a few blocks, find pseudoskeleton approximations for these blocks, and then try to compress the whole "mosaic-skeleton" approximation by the standard Lanczos-like techniques. This strategy evidently leads to a highly concurrent algorithm. The details are left for another paper.

The function $t(r, n)$ defined by (2.3) and its upper estimates play a key role in our proofs. We have presented here a new estimate (Lemma 2.1),

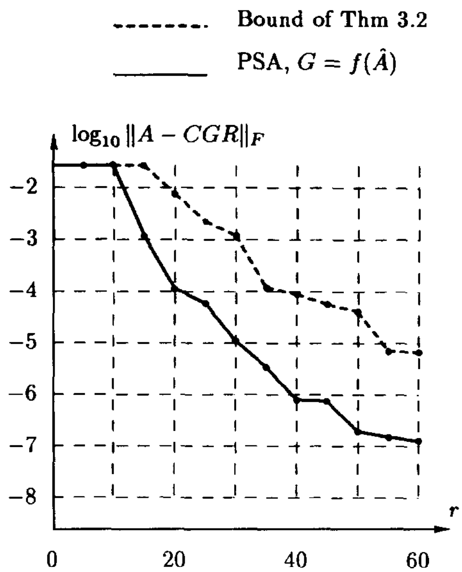

FIG. 4. Upper bounds for approximation error in 2-norm for pseudoskeleton components.

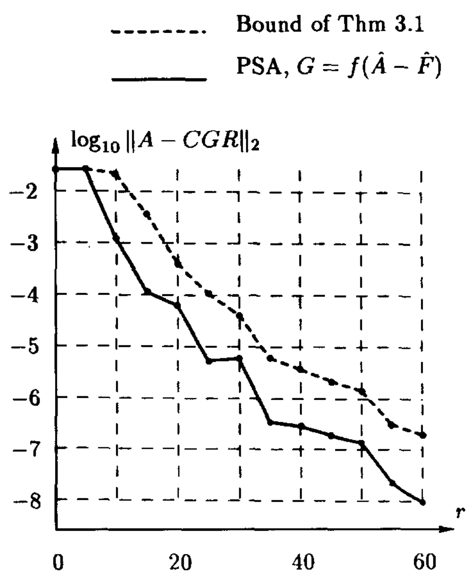

FIG. 5. Upper bounds for approximation error in F-norm for pseudoskeleton components.

TABLE 1
THE MATRIX APPROAIMATION ERROR FOR SVD AND PSEUDOSKELETON COMPONENTS MEASURED IN SPECTRAL NORM

r	SVD	$G = f(A - F)$	$G = f(\hat{A} - \hat{F})$	$G = f(\hat{A})$
0	2.647E-02	2.647E-02	2.647E-02	2.647E-02
5	5.035E-03	7.086E-03	2.647E-02	2.647E-02
10	3.958E-04	8.588E-04	1.227E-03	2.647E-02
15	7.112E-05	8.923E-05	1.131E-04	1.165E-03
20	6.609E-06	1.569E-05	6.242E-05	1.137E-04
25	1.714E-06	4.889E-06	5.351E-06	5.790E-05
30	4.476E-07	1.790E-06	5.892E-06	1.106E-05
35	9.601E-08	3.257E-07	3.457E-07	3.521E-06
40	4.850E-08	9.954E-08	2.864E-07	8.058E-07
45	2.703E-08	7.883E-08	1.944E-07	7.507E-07
50	1.556E-08	3.486E-08	1.383E-07	1.964E-07
55	4.411E-09	1.273E-08	2.307E-08	1.543E-07
60	1.958E-09	6.584E-09	9.814E-09	1.307E-07

which improves the previous one [1, 3]. Finally, we would like to draw the reader's attention to our hypothesis (2.6), since we still have no proof of it.

We would like to thank Gene Golub for good advice about seeking the best-conditioned submatrices. Our special thanks go to Skivkumar Chandrasekaran for kindly sending us the report [1].

TABLE 2
THE MATRIX APPROXIMATION ERROR FOR SVD AND PSEUDOSKELETON COMPONENTS MEASURED IN FROBENIUS NORM

r	SVD	$G = f(A - F)$	$G = f(\hat{A} - \hat{F})$	$G = f(\hat{A})$
0	3.883D-02	3.883D-02	3.883D-02	3.883D-02
5	5.276D-03	8.893E-03	3.883E-02	3.883D-02
10	5.788D-04	1.434E-03	2.287E-03	3.883D-02
15	9.038D-05	1.485E-04	2.246E-04	1.999E-03
20	1.283D-05	3.012E-05	6.914E-05	2.164E-04
25	3.409D-06	8.654E-06	9.177E-06	1.006E-04
30	8.667D-07	2.897E-06	6.423E-06	1.808E-05
35	2.081D-07	5.542E-07	5.842E-07	6.716E-06
40	1.085D-07	2.425E-07	4.326E-07	1.338E-06
45	5.943D-08	1.716E-07	3.229E-07	1.037E-06
50	2.226D-08	6.915E-08	1.570E-07	4.194E-07
55	8.558D-09	2.609E-08	3.585E-08	2.926E-07
60	5.076D-09	1.521E-08	1.785E-08	1.842E-07

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Received 25 April 1994; final manuscript accepted 19 March 1996